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A LIMITING ABSORPTION PRINCIPLE FOR SCHRÖDINGER OPERATORS WITH SPHERICALLY SYMMETRIC EXPLODING POTENTIALS

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ABSTRACT

Let $H = -\Delta + V(r)$ be a Schrödinger operator with a spherically symmetric exploding potential, namely, $V(r) = V_S(r) + V_L(r)$, where $V_S(r)$ is short-range and the exploding part $V_L(r)$ satisfies the following assumptions: (a) $\Lambda = \limsup_{r\to\infty} V_L(r) < \infty$ (but $\Lambda = -\infty$ is possible). Denote $\Lambda^+ = \max(\Lambda, 0)$. (b) $V_L(r) \in C^{2k}(r_0, \infty)$ and, with some $\delta > 0$ such that $2k\delta > 1$: $(d/dr)^j V_L(r) \cdot (\Lambda^+ - V_L(r))^{-1} = O(r^{-j\delta})$ as $r \to \infty$, $j = 1, \cdots, 2k$. (c) $\int_{r_0}^{\infty} dr/|V_L(r)|^{1/2} = \infty$. (d) $(d/dr)V_L(r) \leqq 0$. Under these assumptions a limiting absorption principle for $R(z) = (H - z)^{-1}$ is established. More specifically, if $K \subseteq C^+ = \{z/\operatorname{Im} z \geqq 0\}$ is compact and $K \cap (-\infty, \Lambda] = \varnothing$ then $R(z)$ can be extended as a continuous map of K into $B(Y, Y^*)$ (with the uniform operator topology), where $Y \subseteq L^2(R^n)$ is a weighted-L^2 space. To ensure uniqueness of solutions of $(H - z)u = f$, $z \in K$, a suitable radiation condition is introduced.

I. Introduction

Let $T = -\Delta + V(r)$ be a Schrödinger operator in R^n, $n \geqq 3$. Here V is a real spherically symmetric potential which can be decomposed as the sum of short- and long-range terms:

$$V(r) = V_S(r) + V_L(r), \qquad r = |x|.$$

The following assumptions are made on V:

$$(V1) \qquad V(r) \in L^2(0,\infty)_{\mathrm{loc}};$$

$$(V2) \qquad V(r) = O(r^{-2+\varepsilon}), \quad \varepsilon > 0, \qquad \text{as } r \to 0;$$

$$(V2)' \qquad \int_0^1 V(r)^2 r^{n-1} dr < \infty$$

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($(V2)'$ follows from $(V2)$ for $n \geqq 4$);

$(VL1)$
$$\int_1^\infty \frac{dr}{|V_L(r)|^{1/2}} = \infty;$$

$(VL2)$
$$\limsup_{r\to\infty} V_L(r) = \Lambda < \infty,$$

and with $\Lambda^+ = \max(\Lambda, 0)$, some $\delta > 0$ and some positive integer k such that $2k\delta > 1$, we have:

(i) $V_L(r) \in C^{2k}(r_0, \infty)$,

(ii) $[(d/dr)^j V_L(r)](\Lambda^+ - V_L(r) + 1)^{-1} = O(r^{-j\delta})$ as $r \to \infty$ for $j = 1, \cdots, 2k$,

(VS)
$$V_S(r) = O(r^{-1-\varepsilon}), \quad \varepsilon > 0, \qquad \text{as } r \to \infty.$$

In a previous work [3] it has been shown that, under the above assumptions, T is essentially self-adjoint on $C_0^\infty(R^n)$ and, furthermore, that its spectrum in $\Sigma = (\Lambda, \infty)$ is absolutely continuous. However, in the present study we must impose a further restriction on V_L, as expressed by the following assumption:

(VX) $\quad V_L(r)$ is monotone decreasing in (r_0, ∞), $\quad$ some $r_0 > 0$,

where δ is as in $(VL2)$.

REMARK 1.1. The condition (VS) can be considerably relaxed. The emphasis in this paper is on the role of V_L, and the short-range term is used, in fact, just to facilitate the "smoothing" of V_L by moving local singularities to V_S.

REMARK 1.2. The condition (VX) can be replaced by other conditions, which do not require monotonicity, but are stronger than $(VL2)$(ii). In particular, a condition like that imposed in [5] can be used (which is roughly equivalent to assuming $\delta > \frac{1}{2}$). This will be evident from the proofs.

In this paper we propose to show that the resolvent of H satisfies a "limiting absorption principle" in the neighborhood of Σ. This means, roughly, that the resolvent operator remains bounded "down" to Σ, when taken in a suitable topology. Besides proving the absolute continuity in Σ, this principle establishes a most important tool in the discussion of non-spherically symmetric perturbations, as will be shown in a forthcoming paper.

The limiting absorption principle for Schrödinger operators was first proved by Agmon [1] for the short-range case and by Ikebe–Saito [4] for long-range potentials (with short-range radial derivative). It was then proved for certain types of oscillating potentials by Mochizuki–Uchiyama [6] and for exploding potentials by Jäger–Rejto [5]. A treatment of another class of potentials which do not decay at infinity can be found in [2].

Our class of potentials is similar (with regard to growth properties) to the one discussed by Jäger–Rejto. There are, however, some significant differences, such as:

(a) We allow a growth rate of r^2, whereas in [5] the growth rate is at most r.

(b) In [5], if V_L is bounded then it must necessarily be decaying long-range in the sense of Ikebe–Saito. Here we can deal with potentials which neither explode nor decay at infinity.

(c) Our weighted L^2-estimates involve a weight function which depends on $V_L(r)$. Thus, for a growth rate of r^2, our weight function behaves like $(1+r)^\alpha$, with an arbitrarily small $\alpha > 0$.

Note that, as indicated in [3], a growth rate of r^2 is the maximal possible rate if an absolutely continuous spectrum is still desired.

Our method is based on the representation of T as an ordinary differential operator with operator-valued coefficients, as suggested by Jäger and Saito and summarized in [7].

Finally, let us mention that results somewhat similar to ours were obtained independently by L. Shwartzman, of the Institute of Mathematics at the Hebrew University. His methods are completely different from ours.

II. Notations and preliminaries

The following notations are used throughout the paper: S^{n-1} denotes the unit sphere in R^n; $X = L^2(S^{n-1})$; $R^+ = (0,\infty)$; $L^2(R^+, X)$ denotes the space of X-valued L^2-functions on R^+.

Let $T = -\Delta + V$. It is well-known that by the unitary transformation $U: L^2(R^n) \to L^2(R^+, X)$, given by $Uf(r,\omega) = r^{(n-1)/2} f(r\omega)$, T becomes unitarily equivalent to an operator of the form

$$H = -\frac{d^2}{dr^2} + B(r) + C(r) \tag{2.1}$$

where

$$B(r) = r^{-2}\left(-\tilde{\Delta}_n + \frac{(n-1)(n-3)}{4}\right),$$

$\tilde{\Delta}_n$ is the Laplace–Beltrami operator in X, and $C(r)$ is the multiplication operator by $V(r)$.

Since $V(r)$ is spherically symmetric, we can represent $L^2(R^+, X)$ as a direct sum:

$$L^2(R^+, X) = \sum_j \oplus M_j, \qquad M_j = L^2(R^+), \tag{2.2}$$

such that, for every j, M_j reduces H and

$$H/M_j \simeq -\frac{d^2}{dr^2} + \frac{1}{r^2}\left(\mu_j + \frac{(n-1)(n-3)}{4}\right) + V(r) \tag{2.3}$$

where μ_j is the j-th eigenvalue of $-\tilde{\Delta}_n$.

Let $C_0^\infty(R^+, X)$ be the set of X-valued compactly supported C^∞ functions on R^+. We denote by $CF_0^\infty(R^+, X)$ the subset of $C_0^\infty(R^+, X)$ which consists of functions having only finitely many non-zero components in $\Sigma_j \oplus M_j$. Clearly, $CF_0^\infty(R^+, X)$ is dense in $L^2(R^+, X)$.

Let $(\ ,\ )_X$ be the scalar product in X and $\|\phi\|_X^2 = (\phi, \phi)_X$. Define a norm $|||\phi|||$ on $C_0^\infty(R^+, X)$ by

$$|||\phi|||^2 = \int_0^\infty \left\{ \left\| \frac{d\phi}{dr} \right\|_X^2 + \| B\phi(r)\|_X^2 + \|\phi(r)\|_X^2 \right\} dr.$$

The completion of $C_0^\infty(R^+, X)$ with respect to $|||\ \ |||$ is denoted by $H_0^1(R^+, X)$. Clearly, an element of $H_0^1(R^+, X)$ has a weak derivative in $L^2(R^+, X)$.

Let $I = [\alpha, \beta] \subseteq (\Lambda, \infty)$, where $\Lambda = \limsup_{r\to\infty} V_L(r)$. By rearranging the sum $V = V_S + V_L$ we see that we may assume $\alpha > V_L(r)$ for $r \in R^+$.

For a small $\mu_0 > 0$ set

$$\begin{aligned} \Omega^+(I) &= \{z / \alpha \leqq \operatorname{Re} z \leqq \beta, 0 < \operatorname{Im} z \leqq \mu_0\}, \\ \bar{\Omega}^+(I) &= \{z / \alpha \leqq \operatorname{Re} z \leqq \beta, 0 \leqq \operatorname{Im} z \leqq \mu_0\}. \end{aligned} \tag{2.4}$$

We are interested in solutions of the equation

$$(H - z)u = f \tag{2.5}$$

where H is given by (2.1), $z = \lambda + i\mu \in \bar{\Omega}^+(I)$ and f is suitably restricted in $L^2(R^+, X)$.

By standard regularity theorems a solution u of (2.5) is in $H_0^1(R^+, X)_{\rm loc}$ and u'' exists weakly $L^2(R^+, X)_{\rm loc}$. Thus, (2.5) holds pointwise for a.e. $r \in R^+$.

We now apply a sequence of transformations of the radial variable r and the function u in (2.5), following [3] (taking $\operatorname{Im}\sqrt{z} \geqq 0$ iff $\operatorname{Im} z \geqq 0$):

$$\begin{aligned} &\xi^1 = \int_0^r \sqrt{z - V_L(s)}\,ds, \qquad D^1 = \tfrac{1}{2} V_L'(r)(z - V_L(r))^{-3/2}, \\ &u^1(\xi^1) = u(r(\xi^1, z))(z - V_L(r))^{1/4}. \end{aligned} \tag{2.6}$$

Let

$$
\begin{aligned}
&P_1^1(\xi^1) = -\left[\tfrac{1}{4}V_L''(r)(z - V_L(r))^{-2} + \tfrac{5}{16}V_L'(r)^2(z - V_L(r))^{-3}\right],\\
(2.7)\qquad &P_2^1(\xi^1) = V_S(r)(z - V_L(r))^{-1}, \qquad B^1(\xi^1) = (z - V_L(r))^{-1}B(r).
\end{aligned}
$$

Here r and ξ^1 are related (for a fixed z) by (2.6).

It is now easily verified that u^1 satisfies the equation

$$
(2.8)\qquad \left[-\left(\frac{d}{d\xi^1}\right)^2 + B_1^1(\xi^1) + P_1^1 + P_2^1 - 1\right]u^1(\xi^1) = f^1(\xi^1)
$$

with

$$
f^1(\xi^1) = (z - V_L(r))^{-3/4}f(r).
$$

Note that here the differentiation is along the curve traced out by $\xi^1 = \xi^1(r, z)$ in the complex plane.

For $j = 2, \cdots, k$ we now define successively

$$
\xi^j = \int_0^{\xi^{j-1}} \sqrt{1 - P_1^{j-1}(s)}\,ds, \quad D^j(\xi^j) = \frac{1}{2}\frac{d}{d\xi^{j-1}}P_1^{j-1}(\xi^{j-1})(1 - P_1^{j-1}(\xi^{j-1}))^{-3/2},
$$

$$
P_1^j(\xi^j) = -\frac{1}{2}\frac{d}{d\xi^j}D^j(\xi^j) + \frac{1}{4}[D^j(\xi^j)]^2, \quad P_2^j = P_2^{j-1}(\xi^{j-1})(1 - P_1^{j-1}(\xi^{j-1}))^{-1},
$$

$$
B^j(\xi^j) = B^{j-1}(\xi^{j-1})(1 - P_1^{j-1}(\xi^{j-1}))^{-1}, \quad u^j(\xi^j) = u^{j-1}(\xi^j)(1 - P_1^{j-1}(\xi^{j-1}))^{1/4}.
$$

As shown in [3], the functions $P_1^j(\xi^j(r, z))$ satisfy the following estimates:

$$
\begin{aligned}
&P_1^j(\xi^j(r, z)) = O(|z - V_L(r)|^{-1}r^{-2j\delta}),\\
(2.9)\qquad &\frac{d}{dr}P_1^j(\xi^j(r, z)) = O(|z - V_L(r)|^{-1}r^{-(2j+1)\delta}),
\end{aligned}
$$

uniformly in $z \in \tilde{\Omega}^+(I)$, as $r \to \infty$ and $j = 1, \cdots, k$ (for $j = k$ we have only the first estimate).

Set

$$
\begin{aligned}
&\xi = \xi^k, \quad P_1 = P_1^k, \quad P_2 = P_2^k, \quad F = B^k, \quad P = P_1 + P_2,\\
(2.10)\qquad &Q(r, z) = \sqrt{z - V_L(r)}\prod_{l=1}^{k-1}(1 - P_1^l(\xi^l(r, z)))^{1/2}.
\end{aligned}
$$

Thus, ξ and r are related by

$$
(2.11)\qquad d\xi = Q(r, z)dr, \qquad \operatorname{Im} Q \geqq 0
$$

and, with

(2.12) $$v = u^k = \sqrt{Q}u, \qquad g = Q^{-3/2}f,$$

equation (2.5) takes the form

(2.13) $$Rv \equiv (-d^2/d\xi^2 + F + P - 1)v = g.$$

In what follows we shall use either ξ or r, keeping (2.11) in mind.

Note that by (2.9) and assumptions (VS) and ($VL2$) we have

(2.14) $$P(r,z) = O(|z - V_L(r)|^{-1} r^{-(1+\gamma)})$$

as $r \to \infty$, uniformly in $z \in \tilde{\Omega}^+(I)$, where $\gamma = \min(2k\delta - 1, \varepsilon)$.

Also, by the construction

(2.15) $$F(r,z) = Q^{-2}B(r) = (rQ)^{-2}\left(-\tilde{\Delta}_n + \frac{(n-1)(n-3)}{4}\right).$$

Finally, we introduce the weighted $L^2_{\beta,\omega}(R^+, X)$ spaces: Let $\omega(r) > 0$ be a continuous weight function and let β be a real number. Then

(2.16) $$L^2_{\beta,\omega}(R^+, X) = \Big\{ u/u \in L^2(R^+, X)_{\mathrm{loc}}, \quad \|u\|^2_{\beta,\omega} = \int_0^\infty (1+r)^{2\beta}\omega(r)\|u(r)\|^2_X dr < \infty \Big\}.$$

When $\omega(r) \equiv 1$, the index ω will be suppressed: $L^2_\beta(R^+, X) \equiv L^2_{\beta,1}(R^+, X)$, and $\|u\|_\beta \equiv \|u\|_{\beta,1}$.

III. The limiting absorption principle

In this section we study the behavior of solutions of (2.5) or, equivalently, (2.13):

(3.1) $$Rv = g.$$

We assume that $z = \lambda + i\mu \in \tilde{\Omega}^+(I)$, $I = [\alpha, \beta]$ with $\alpha > \limsup_{r\to\infty} V_L(r)$.

Following Saito [7] we call a solution of (3.1) a radiative function, if it satisfies a suitable radiation condition.

Let

(3.2) $$\tfrac{1}{2} < \sigma < \min(\tfrac{1}{2} + \varepsilon, \tfrac{1}{2} + \delta/2, 2k\delta - \tfrac{1}{2})$$

where ε and δ are as in assumptions (VS), ($VL2$) respectively.

Definition 3.1. Let $g \in L^2(R^+, X)_{\mathrm{loc}}$, $z \in \tilde{\Omega}^+(U)$. A function $v(\xi) \in$

$H_0^1(R^+,X)_{\mathrm{loc}}$ is called *a radiative function* for (g,z), if it satisfies the following conditions:

v solves (3.1) in the weak sense, namely, for $\phi \in C_0^\infty(R^+,X)$,

$$\int_0^\infty (v(\xi), R^*\phi)_X d\xi = \int_0^\infty (g(\xi),\phi(\xi))_X d\xi; \tag{3.3}$$

$$dv/d\xi - iv \in L^2_{\sigma-1}(R^+,X). \tag{3.4}$$

We denote $v = \mathrm{rad}(g,z)$.

REMARK 3.2. Using (2.11) we see that (3.4) can be written explicitly as

$$\int_0^\infty |Q|^{-2}(1+r)^{2(\sigma-1)}\left\|\frac{dv}{dr} - iQv\right\|_X^2 dr < \infty. \tag{3.5}$$

Of course, the radiation condition can be written in terms of a solution of the original equation (2.5). However, it will be seen that (3.4)–(3.5) are technically advantageous, until some estimates are established and a simplified version is obtained (see Theorem 3.6).

The following two lemmas deal with the uniqueness and the existence of the radiative function when g is suitably restricted.

LEMMA 3.3. (Uniqueness). *Let v be a radiative function for $(0,z)$. Then $v \equiv 0$.*

LEMMA 3.4 (Existence). *Let $g \in CF_0^\infty(R^+,X)$. Then $v = \mathrm{rad}(g,z)$ exists, for every $z \in \tilde{\Omega}^+(I)$.*

In particular, if $\mathrm{Im}\, z > 0$, $f = Q^{3/2}g$ and $u = (H-z)^{-1}f$, then $v = \sqrt{Q}u$.

PROOF OF LEMMA 3.3. Let v_j be the component of v in M_j (see (2.2)). By the results obtained in [3] v_j is a linear combination of two fundamental solutions ϕ_1, ϕ_2 having the following asymptotic behavior:

$$\begin{aligned} \phi_1(\xi) &= e^{i\xi}(1+a_1(\xi)), & \frac{d\phi_1}{d\xi}(\xi) &= ie^{i\xi}(1+a_2(\xi)), \\ \phi_2(\xi) &= e^{-i\xi}(1+a_3(\xi)), & \frac{d\phi_2}{d\xi}(\xi) &= -ie^{-i\xi}(1+a_4(\xi)) \end{aligned} \tag{3.6}$$

(ξ determined by (2.11)) where

$$\sum_{i=1}^4 |a_i(\xi(r,z))| = O(|z - V_L(r)|^{-1/2} r^{-\tau}) \quad \text{as } r \to \infty, \quad \tau = \min(\delta, 2k\delta - 1); \tag{3.7}$$

using $|e^{i\xi}| \leqq 1$ and (3.7) we see that ϕ_1 satisfies (3.4) (more precisely, since ϕ_1

may be singular near $r = 0$, it must be replaced by $\gamma\phi_1$, where γ is a suitable cutoff function near the origin), whereas ϕ_2 does not. Hence there exists a constant d such that

$$v_j = d\phi_1. \tag{3.8}$$

As a radiative function $v_j \in H_0^1(R^+)_{loc}$ so that (3.8) implies

$$\phi_1(0) = 0. \tag{3.9}$$

If $\mu = \operatorname{Im} z > 0$ then by (3.6) ϕ_1 is exponentially decaying and together with (3.9) we see that $Q^{-\frac{1}{2}}\phi_1$ is an eigenfunction of H, with a non-real eigenvalue (z), which is a contradiction.

Assume now that $\mu = 0$, and $d \neq 0$. Then ξ is a real variable and ϕ_1 solves a homogeneous ordinary differential equation of the type (2.13) ($g = 0$). It is well-known that in this case the Wronskian of ϕ_1 is independent of ξ. The condition (3.9) implies that ϕ_1 is a scalar multiple of a real solution, hence

$$W(\phi_1, \bar{\phi}_1) \equiv \text{real}. \tag{3.10}$$

On the other hand, using (3.6) we get

$$W(\phi_1, \bar{\phi}_1) = \lim_{\xi \to \infty} \left(\phi_1(\xi) \overline{\frac{d\phi_1}{d\xi}} - \overline{\phi_1(\xi)} \frac{d\phi_1}{d\xi} \right) = -2i$$

which contradicts (3.10). Thus $d = 0$ and the lemma follows. Q.E.D.

PROOF OF LEMMA 3.4. Let $H_j = H/M_j$ (2.3) and, for $z \in \Omega^+(I)$, let $K_j(x, y; z)$ be its resolvent kernel. The existence of K_j and its continuity on $\bar{R}^+ \times \bar{R}^+ \times \bar{\Omega}^+(I)$ were proved in [3]. Let $f_j = Q^{3/2} g_j$, where g_j is the component of g in M_j. Let

$$u_j(r, z) = \int_0^\infty K_j(r, s; z) f_j(s) ds \tag{3.11}$$

and set $v_j(r, z) = \sqrt{Q}\, u_j(r, z)$.

The integration in (3.11) extends over the support of f_j. If $r > \sup\operatorname{supp} f_j$ then

$$v_j(r(\xi, z), z) = d\phi_1(\xi) \tag{3.12}$$

where ϕ_1 is given by (3.6).

Obviously, v_j and g_j satisfy (3.3). The validity of (3.4) follows from (3.12) and the previous proof.

Since there is only a finite number of non-zero components g_j the lemma is established. Q.E.D.

We turn now to the formulation of the limiting absorption principle in the transformed spaces:

THEOREM 3.5. *Let* $z=\lambda+i\mu\in\tilde{\Omega}^+(I)$ *and* $g\in L^2_{\sigma,|Q|^2}(R^+,X)$. *Then*:

(a) *The radiative function* $v=\operatorname{rad}(g,z)$ *exists and is unique. Furthermore,*

$$v\in L^2_{-\sigma}(R^+,X). \tag{3.13}$$

(b) *There exists a constant C, which depends only on* $\tilde{\Omega}^+(I)$ *(and the operator), such that*

$$\|v\|_{-\sigma}+\left\|\frac{dv}{d\xi}-iv\right\|_{\sigma-1}+\|B^{1/2}v\|_{\sigma-1,|Q|^{-2}}\leqq C\|g\|_{\sigma,|Q|^2}. \tag{3.14}$$

(c) *The mapping* $(g,z)\to\operatorname{rad}(g,z)$ *is continuous as a mapping of*

$$L^2_{\sigma,|Q|^2}(R^+,X)\times\tilde{\Omega}^+(I)\to L^2_{-\sigma}(R^+,X). \tag{3.15}$$

Furthermore, if $Y(z)g=\operatorname{rad}(g,z)$ *then the mapping*

$$z\to Y(z)\in B(L^2_{\sigma,|Q|^2}(R^+,X),L^2_{-\sigma}(R^+,X)) \tag{3.16}$$

is continuous, where the uniform topology is employed for the operator space.

PROOF. In the process of the proof, C will denote a generic constant which depends only on $\tilde{\Omega}^+(I)$ and the operator. Observe that by (2.14)

$$P(r,z)=O(|Q|^{-2}r^{-2\sigma})\qquad\text{as } r\to\infty. \tag{3.17}$$

We shall prove the theorem in several steps.

Step A. Take $g\in CF_0^\infty(R^+,X)$. Lemmas 3.3, 3.4 imply the existence and uniqueness of $v=\operatorname{rad}(g,z)$ and the fact that $v\in L^2_{-\sigma}(R^+,X)$ follows from (3.6), (3.12) and $\sigma>\frac{1}{2}$. We now prove that

$$\left\|\frac{dv}{d\xi}-iv\right\|_{\sigma-1}+\|B^{1/2}v\|_{\sigma-1,|Q|^{-2}}\leqq C(\|v\|_{-\sigma}+\|g\|_{\sigma,|Q|^2}). \tag{3.18}$$

To show (3.18), rewrite (3.1) as

$$-\frac{d}{d\xi}\left(\frac{dv}{d\xi}-iv\right)-i\left(\frac{dv}{d\xi}-iv\right)+Fv+Pv=g. \tag{3.19}$$

Multiply (3.19) by Q and use (2.11) to get

$$-\frac{d}{dr}\left(\frac{dv}{d\xi}-iv\right)-iQ\left(\frac{dv}{d\xi}-iv\right)+\frac{B}{Q}v+Jv=Qg \tag{3.20}$$

where, by (3.17),

(3.21) $$J = O(r^{-2\sigma}) \quad \text{as } r \to \infty, \quad \text{uniformly in } z \in \tilde{\Omega}^+(I).$$

Taking the real part of the scalar product (in X) of (3.20) with $dv/d\xi - uv$ and denoting $v' = dv/d\xi$ we obtain

(3.22) $$-\frac{1}{2}\frac{d}{dr}\|v' - iv\|_X^2 + (\operatorname{Im} Q)\|v' - iv\|_X^2 + \operatorname{Re}\left(\frac{B}{Q}v, v' - iv\right)_X + \operatorname{Re}(Jv, v' - iv)_X = \operatorname{Re}(gQ, v' - iv)_X$$

so that

(3.23) $$\operatorname{Im} Q \geqq 0, \qquad r \geqq r_0$$

where r_0 is independent of $z \in \tilde{\Omega}^+(I)$. Also

(3.24) $$\operatorname{Re}\frac{1}{Q}(Bv, -iv)_X = |Q|^{-2}(-\operatorname{Im}\bar{Q})(Bv, v)_X \geqq 0$$

where the positivity of B has been used;

(3.25) $$\operatorname{Re}\frac{1}{Q}(Bv, v')_X = |Q|^{-2}\left[\frac{1}{2}\frac{d}{dr}\|B^{1/2}v\|_X^2 + \frac{1}{r}\|B^{1/2}v\|_X^2\right]$$

where we have used that $dB/dr = -(2/r)B$.

Combining (3.23)–(3.25) with (3.22) we have

(3.26) $$-\frac{1}{2}\frac{d}{dr}\|v' - iv\|_X^2 + |Q|^{-2}\left[\frac{1}{2}\frac{d}{dr}\|B^{1/2}v\|_X^2 + \frac{1}{r}\|B^{1/2}v\|_X^2\right] \leqq |(Jv, v' - iv)_X| + |(gQ, v' - iv)_X|.$$

Multiply (3.26) by $(1+r)^{2\sigma-1}$ and integrate from r_0 to r:

(3.27) $$\begin{aligned}&\int_{r_0}^{r}\left(\sigma - \frac{1}{2}\right)(1+s)^{2(\sigma-1)}\|v'(s) - iv(s)\|_X^2 ds \\ &+ \int_{r_0}^{r}|Q|^{-2}\left(\frac{1}{s} - \frac{\sigma - \frac{1}{2}}{s+1}\right)(1+s)^{2\sigma-1}\|B^{1/2}v(s)\|_X^2 ds \\ &- \frac{1}{2}\int_{r_0}^{r}\left(\frac{d}{ds}|Q|^{-2}\right)\|B^{1/2}v\|_X^2 ds \\ &\leqq \frac{1}{2}(1+r)^{2\sigma-1}\|v' - iv\|_X^2 + (1+r_0)^{2\sigma-1}\|B^{1/2}v(r_0)\|_X^2|Q(r_0, z)|^{-2} \\ &+ C(\|v\|_{-\sigma}\|v' - iv\|_{\sigma-1} + \|g\|_{\sigma,|Q|^2}\|v' - iv\|_{\sigma-1})\end{aligned}$$

where (3.21) and $-\sigma < \sigma - 1$ were used to estimate $(Jv, v' - iv)_X$. Since $v' - iv \in L^2_{\sigma-1}(R^+, X)$ and noting assumption (VX) it follows that

$$\liminf_{r\to\infty} (1+r)^{2\sigma-1} \| v' - iv \|_X^2 = 0, \qquad \frac{d}{ds} |Q|^{-2} \leqq 0. \tag{3.28}$$

Using a standard trace estimate for an H^2_{loc} function (namely, that the H^1 norm of a function on a compact manifold can be estimated in terms of its H^2 norm in a neighborhood of the manifold) and $|ab| \leqq \eta a^2 + (1/\eta) b^2$ we have

$$\begin{aligned} &(1+r_0)^{2\sigma-1} \| B^{1/2} v(r_0) \|_X^2 \leqq C(\| v \|_{-\sigma} + \| g \|_{\sigma,|Q|^2}),\\ &\| v \|_{-\sigma} \| v' - iv \|_{\sigma-1} + \| g \|_{\sigma,|Q|^2} \| v' - iv \|_{\sigma-1} \\ &\qquad \leqq \eta \| v' - iv \|^2_{\sigma-1} + C(\eta)(\| v \|^2_{-\sigma} + \| g \|^2_{\sigma,|Q|^2}); \end{aligned} \tag{3.29}$$

(3.27) combined with (3.28)–(3.29) for η small enough yields (3.18).

Step B. Assuming again $g \in CF_0^\infty(R^+, X)$ and using the same notation as in step A we now show

$$\int_r^\infty \| v(s) \|_X^2 (1+s)^{-2\sigma} ds \leqq C r^{-(2\sigma-1)} (\| v \|^2_{-\sigma} + \| g \|^2_{\sigma,|Q|^2}). \tag{3.30}$$

To show (3.30) we start from

$$\left\| \frac{dv}{d\xi} - iv \right\|_X^2 = \left\| \frac{dv}{d\xi} \right\|_X^2 + \| v \|_X^2 - 2\,\mathrm{Im}\left(\frac{dv}{d\xi}, v\right)_X. \tag{3.31}$$

To evaluate $(dv/d\xi, v)_X$ we proceed as follows. Take the scalar product (in X) of (3.1) with v:

$$\left(-\frac{d^2 v}{d\xi^2}, v\right)_X + (Fv, v)_X + (Pv, v)_X - \| v \|_X^2 = (g, v)_X. \tag{3.32}$$

Integrate (3.32) along the curve traced out by $\xi = \xi(r, z)$ and take the imaginary part to get

$$\begin{aligned} &\mathrm{Im} \int_0^\xi \left(-\frac{d^2 v}{d\eta^2}, v\right)_X d\eta + \mathrm{Im} \int_0^\xi [(Fv, v)_X + (Pv, v)_X - \| v \|_X^2] d\eta \\ &\quad = \mathrm{Im} \int_0^\xi (g, v)_X d\eta. \end{aligned} \tag{3.33}$$

But

$$\int_0^\xi \left(-\frac{d^2 v}{d\eta^2}, v\right)_X d\eta = \int_0^\xi \left(\frac{dv}{d\eta}, \frac{dv}{d\eta}\right)_X d\eta - \left(\frac{dv}{d\eta}, v\right)_X (\eta = \xi).$$

Hence

$$\text{(3.34)}\qquad \begin{aligned} \operatorname{Im}\left(\frac{dv}{d\xi}, v\right)_X &= \operatorname{Im}\left\{\int_0^\xi \left[\left(\frac{dv}{d\eta}, \frac{\overline{dv}}{d\eta}\right)_X + (Fv, v)_X - \| v \|_X^2\right] d\eta\right\} \\ &\quad + \operatorname{Im}\left\{\int_0^\xi [(Pv, v)_X - (g, v)_X] d\eta\right\} \\ &= I_1 + I_2. \end{aligned}$$

Using (2.11), (2.15)

$$I_1 = \operatorname{Im}\left\{\int_0^r \left(\frac{1}{Q}\left\|\frac{dv}{ds}\right\|_X^2 + \frac{1}{Q^2}\| B^{1/2} v \|_X^2 - \| v \|_X^2\right) ds\right\},$$

and since $\operatorname{Im} 1/Q \leqq 0$ we get

$$\text{(3.35)}\qquad I_1 \leqq 0.$$

I_2 can be estimated easily, using (3.17):

$$\text{(3.36)}\qquad I_2 \leqq C(\| v \|_{-\sigma}^2 + \| g \|_{\sigma,|Q|^2} \| v \|_{-\sigma}).$$

Combining (3.34)–(3.36) with (3.31)

$$\text{(3.37)}\qquad \| v \|_X^2 \leqq \left\|\frac{dv}{d\xi} - iv\right\|_X^2 + C(\| v \|_{-\sigma}^2 + \| g \|_{\sigma,|Q|^2} \| v \|_{-\sigma}).$$

Multiply (3.37) by $(1+s)^{-2\sigma}$ and integrate from r to ∞:

$$\begin{aligned} \int_r^\infty (1+s)^{-2\sigma} \| v(s) \|_X^2 ds \leqq C\Big\{ &(1+r)^{-2(2\sigma-1)} \left\|\frac{dv}{d\xi} - iv\right\|_{\sigma-1}^2 \\ &+ (1+r)^{-(2\sigma-1)}[\| v \|_{-\sigma}^2 + \| g \|_{\sigma,|Q|^2}^2]\Big\}. \end{aligned}$$

Applying (3.18) we get (3.30).

Step C. We now prove (3.14) for $g \in CF_0^\infty(R^+, X)$. In view of (3.18) it suffices to show that

$$\text{(3.38)}\qquad \| v \|_{-\sigma} \leqq C \| g \|_{\sigma,|Q|^2}.$$

Indeed, assume that (3.38) does not hold. This means that we can find sequences

$$\{g_n\} \subseteq CF_0^\infty(R^+, X), \quad \{z_n\} \subseteq \tilde{\Omega}^+(I), \quad \{v_n = \operatorname{rad}(g_n, z_n)\}$$

such that

$$\text{(3.39)}\qquad \| g_n \|_{\sigma,|Q|^2} = \frac{1}{n}, \qquad \| v_n \|_{-\sigma} = 1.$$

We can assume that $z_n \to z$.

Standard elliptic estimates imply that $\{v_n\}$ is bounded in $H_0^1(R^+, X)_{\mathrm{loc}}$. Hence by the Rellich compactness theorem there exists a subsequence $\{v_{n_m}\}$ such that

$$v_{n_m} \to v \quad \text{in } L^2(R^+, X)_{\mathrm{loc}}$$

which, together with (3.30) (applied to v_{n_m}), yields

$$v = \lim_{m \to \infty} v_{n_m} \qquad \text{in } L^2_{-\sigma}(R^+, X). \tag{3.40}$$

Elliptic estimates imply further

$$\left.\begin{aligned} \frac{dv_{n_m}}{dr} &\to \frac{dv}{dr} \\ B^{1/2} v_{n_m} &\to B^{1/2} v \end{aligned}\right\} \quad \text{in } L^2(R^+, X)_{\mathrm{loc}}. \tag{3.41}$$

Applying (3.18) to v_{n_m} and passing to the limit $m \to \infty$ (on a sequence of increasing intervals) we get

$$\left\| \frac{dv}{d\xi} - iv \right\|_{\sigma - 1} < \infty;$$

v is obviously a solution of (3.3) with $g = 0$. Thus $v = \mathrm{rad}(0, z)$ and Lemma 3.3 implies $v = 0$, which is in contradiction to (3.39)–(3.40).

Step D. We now conclude the proof of the theorem by showing the continuity of the mapping (3.15) when g is restricted to $CF_0^\infty(R^+, X)$. The proof implies the existence of $\mathrm{rad}(g, z)$ for a general $g \in L^2_{\sigma, |Q|^2}(R^+, X)$ by a suitable limiting process.

Let $\{g_n\} \subseteq CF_0^\infty(R^+, X)$ be a Cauchy sequence in $L^2_{\sigma,|Q|^2}(R^+, X)$, let $z_n \to z$ in $\Omega^+(I)$ and let $v_n = \mathrm{rad}(g_n, z_n)$. By (3.38) $\{v_n\}$ is bounded in $L^2_{-\sigma}(R^+, X)$ and, by interior elliptic estimates, it is bounded also in $H_0^1(R^+, X)_{\mathrm{loc}}$. Thus, by the same argument leading to (3.40) we obtain the existence of a subsequence $\{v_{n_m}\}$ such that

$$v = \lim_{m \to \infty} v_{n_m} \qquad \text{in } L^2_{-\sigma}(R^+, X)$$

and as in the argument following (3.41) we infer

$$v = \mathrm{rad}(g, z), \quad g = \lim_{m \to \infty} g_n \qquad \text{in } L^2_{\sigma,|Q|^2}(R^+, X). \tag{3.42}$$

Since every subsequence of $\{v_n\}$ has a subsequence converging to v, the uniqueness of the radiative function now implies

$$\lim_{n\to\infty} \operatorname{rad}(g_n, z_n) = \operatorname{rad}(g, z) \quad \text{in } L^2_{-\sigma}(R^+, X). \tag{3.43}$$

Finally, to prove the continuity of the mapping (3.16) we note that (3.43) holds even in the case that $g_n \to g$ only weakly in $L^2_{\sigma,|Q|^2}(R^+, X)$. Indeed, an inspection of the argument leading to (3.43) shows that all that is needed is the convergence of the right-hand side of (3.3) (g replaced by g_n) and the boundedness of $\{g_n\}$, which is guaranteed by the weak convergence too.

Therefore, to prove (3.16) assume, to the contrary, that there exist sequences

$$\{g_n\} \subseteq L^2_{\sigma,|Q|^2}(R^+, X), \qquad \|g_n\|_{\sigma,|Q|^2} = 1, \quad z_n \to z \quad \text{in } \tilde{\Omega}^+(I)$$

such that

$$\|(Y(z_n) - Y(z))g_n\|_{-\sigma} \geqq \eta > 0, \qquad \text{all } n. \tag{3.44}$$

Now, if we extract a weakly convergent subsequence $\{g_{n_m}\}$ and use (3.43) and the remarks thereafter we get a contradiction to (3.44). Q.E.D.

We now formulate the limiting absorption principle for the original operator in R^n.

THEOREM 3.6. *Let $T = -\Delta + V$ be a Schrödinger operator in R^n, where V satisfies the requirements listed in the Introduction. Let $I = [\alpha, \beta]$ be a real interval where*

$$\alpha > \limsup_{r\to\infty} V_L(r) = \Lambda$$

and set

$$\Omega^+(I) = \{z / \alpha \leqq \operatorname{Re} z \leqq \beta, 0 < \operatorname{Im} z \leqq \mu\},$$

$$\tilde{\Omega}^+(I) = \text{closure of } \Omega^+(I);\ \Lambda^+ = \max(\Lambda, 0).$$

Let $R(z) = (T - z)^{-1}$, $z \in \Omega^+(I)$. Let

$$\omega(r) = (\Lambda^+ + 1 - V_L(r))^{1/2} \tag{3.45}$$

and $\frac{1}{2} < \sigma < \min(\frac{1}{2} + \delta/2, \frac{1}{2} + \varepsilon, 2k\delta - \frac{1}{2})$. Define, for s real and $\eta = \eta(r) > 0$,

$$\|f\|^2_{s,\eta} = \int_{R_n} (1 + |x|)^{2s}\eta(|x|)|f(x)|^2 dx, \qquad L^2_{s,\eta}(R^n) = \{f / \|f\|_{s,\eta} < \infty\}.$$

Then:

(a) *$R(z)$ is bounded as an operator from $L^2_{\sigma,\omega^{-1}}(R^n)$ into $L^2_{-\sigma,\omega}(R^n)$, uniformly in $z \in \Omega^+(I)$.*

(b) *$R(z)$ can be extended as a continuous map $\tilde{R}(z)$ of $\tilde{\Omega}^+(I)$ into $B(L^2_{\sigma,\omega^{-1}}, L^2_{-\sigma,\omega})$ equipped with the operator norm.*

(c) *Let $z \in \tilde{\Omega}^+(I)$, $f \in L^2_{\sigma,\omega^{-1}}(R^n)$.*

Let $u = \tilde{R}(z)f$. Then u satisfies the following "Radiation Condition":

$$\int_{R^n} \omega(|x|)^{-1}(1+|x|)^{2(\sigma-1)}\left|\frac{\partial u}{\partial |x|} - i\sqrt{z - V_L(|x|)}u\right|^2 dx \leqq C\int_{R^n}(1+|x|)^{2\sigma}\omega(|x|)^{-1}|f(x)|^2 dx. \tag{3.46}$$

C depends only on $\tilde{\Omega}^+(I)$ and the operator.

PROOF. Note that by (2.10) and (3.45) there exists a constant c, depending on $\tilde{\Omega}^+(I)$ only, such that

$$c^{-1}|Q(r,z)| \leqq \omega(r) \leqq c\,|Q(r,z)|, \qquad z \in \tilde{\Omega}^+(I).$$

Combined with (2.12) we see that (3.38) can be written as

$$\int_{R^n} \omega(|x|)(1+|x|)^{-2\sigma}|u(x)|^2 dx \leqq C\int_{R^n} \omega(|x|)^{-1}(1+|x|)^{2\sigma}|f(x)|^2 dx. \tag{3.47}$$

The continuity properties of $\tilde{R}(z)$ now follow easily from Theorem 3.5. As for (3.46), note that (2.12) and (3.14) imply

$$\int_0^\infty \omega(r)^{-2}(1+r)^{2(\sigma-1)}\int_{S^{n-1}}\left|\frac{\partial}{\partial r}(r^{(n-1)/2}\sqrt{Q}u) - iQ^{3/2}r^{(n-1)/2}u\right|^2 dS dr \leqq C\int_{R^n}\omega(|x|)^{-1}(1+|x|)^{2\sigma}|f(x)|^2 dx. \tag{3.48}$$

But

$$|Q|(1+r)^{2(\sigma-1)}\left|\frac{d}{dr}(r^{(n-1)/2})\right|^2 \leqq c\omega(r)r^{n-1}(1+r)^{-2\sigma} \tag{3.49}$$

and also, by assumption ($VL2$),

$$\frac{1}{Q}\frac{dQ}{dr} = O(r^{-\delta}), \qquad r \to \infty$$

so

$$(1+r)^{2(\sigma-1)}\left|\frac{d\sqrt{Q}}{dr}\right|^2 \leqq c\omega(r)(1+r)^{-2\sigma}. \tag{3.50}$$

Finally, writing (2.1) as

$$Q(r,z)=\sqrt{z-V_{\mathrm{L}}(r)}(1+a(r,z))^{1/2}$$

and using the estimates (2.9) we have

$$|z-V_{\mathrm{L}}(r)|\cdot|a(r,z)|\cdot(1+r)^{2(\sigma-1)}\leqq c\omega(r)(1+r)^{-2\sigma}. \tag{3.51}$$

Estimates (3.49)–(3.51), combined with (3.47)–(3.48), now yield (3.46). Q.E.D.

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